The Simplex Algorithm – Option 2

A Sentry Reconnect Module

- Note: This module will have a professional development module available on Canvas by the end of 2023

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The simplex algorithm is a common method of solving linear programs. This module will define common terms used for linear programs; provide a step by step explanation of the method; provide a two dimensional graphical interpretation of the method; and demonstrate a free software tool that replicates the simplex method.

In linear algebra, the goal is to find the point that satisfies a set of equations. In linear programs, we have a set of equations that form a feasible space. Any point within that space satisfies those constraints. Another function is introduced where the goal is to maximize or minimize its value. This function is called the objective. So, the goal is to find the best solution from that set of solutions in the feasible space.

Linear programs are the set of optimization problems where all of the variable exponents are 1. This would be lines in 2-d, surfaces in 3-d and higher dimension. A linear program has a set of decision variables whose values we are trying to determine based on a set of constraint equations (boundaries for the problem) and an objective function which is being maximized or minimized.

The Algebra of a Linear Program

Imagine this simple example

A company manufactures two products, A and B. Product A sells for $10 of profit and product B sells for $12. The company has three departments which the products pass through. Department 1 requires 4 hours for each product A and product B. There are 600 hours of labor available in this department. Department 2 requires 3 hours for each product A and 2 hours for a product B. This department has 500 hours available. Department 3 requires 2 hours for each product A and 4 hours for product B. There are 500 hours available in this department

Department	Hours / Product A	Hours / Product B	Available hours
1	4	4	600
2	3	2	500
3	2	4	500

If x_a and x_b are the number of products A and B to be manufactured. The total profit (which we will call z is would be $10x_a$ and $12x_b$. But the process has a set of requirements, referred to as constraints. The hours of labor available in each department can not be violated.

Department 1

$$4x_a + 4x_b \le 600$$

For department 2

$$3x_a + 2x_b \le 500$$

For department 3

$$2x_a + 4x_b \le 500$$

The algebra for this is…

$$\text{maximize } z = 10x_a + 12x_b$$

Subject to

$$4x_a + 4x_b \le 600$$

$$3x_a + 2x_b \le 500$$

$$2x_a + 4x_b \le 500$$

$$x_a, x_b \ge 0$$

Let's start with some definitions. A linear program is in standard form if: it is a minimization problem; with equality constraints; and all variables are non-negative.

Minimize $z(x) = d + c^T x$ where d is a constant

Subject to

Ax=b

x>=0

How is the production problem converted to standard form?

Converting from maximization to minimization:

Maximizing $c^T x$ is equivalent to minimizing $-c^T x$ subject to x being feasible in both cases. Keep in mind, $y = -c^T x$ is a reflection of $y = c^T x$, so a maximum in one becomes a minimum in the other.

Converting the inequalities to equality constraints.

For each inequality constraint, a slack variable will be introduced. This slack variable must also be non-negative values. The slack variables will be *u, v* and *w*.

The inequality $4x_a + 4x_b \le 600$ becomes
$4x_a + 4x_b + u = 600$

The inequality $3x_a + 2x_b \le 500$ becomes
$3x_a + 2x_b + v = 500$

The inequality $2x_a + 4x_b \le 500$ becomes
$2x_a + 4x_b + w = 500$

$x_a, x_b, u, v, w \ge 0$

A good practice when solving linear programs is to have all variables on the left side of the equation and constants on the right side.

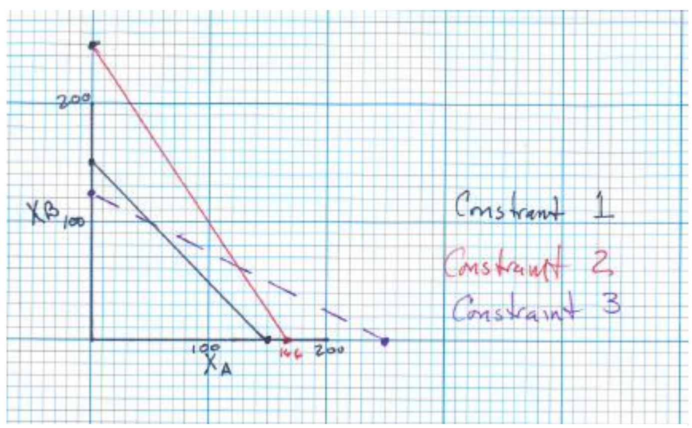
Converting to standard form, the production problem has become

minimize $-z = -10x_a - 12x_b + 0u + 0v + 0w$

$4x_a + 4x_b + u = 600$

$3x_a + 2x_b + v = 500$

$2x_a + 4x_b + w = 500$



A graph showing the boundary lines of the constraints.

And using linear algebra and matrix notations, this would look like…

Minimize z(x) = d + c^Tx where d is a constant

Subject to

Ax=b

x>=0

where d=0

These matrices are:

A – the set of coefficients of the constraints

x_a	x_b	u	v	w
4	4	1	0	0

3	2	0	1	0
2	4	0	0	1

b - the constants for each constraint

600
500
500

c – the coefficients of the objective

-10
-12
0
0
0

d is the constant from the objective function

At this point we form the simplex tableau

cT	"-d"
A	b

eqn 4	-10	-12	0	0	0	0
eqn 1	4	4	1	0	0	600
eqn 2	3	2	0	1	0	500
eqn 3	2	4	0	0	1	500

New definition – Each constraint contains one variable with a coefficient of 1 that does not appear in any other constraint or the objective. Each variable that meets this condition is called a basic variable. The others are called non-basic variables. For this problem, x_a and x_b are non-basic and u, v and w are basic variables. If basic row operations can not be used to get the problem into canonical form, then the problem is infeasible.

If the non-basic variables are set to 0, then the following feasible solution can be determined by inspection.

$$u = 600$$
$$v = 500$$
$$w = 500$$
$$x_a = x_b = 0$$

And the objective value is 0. This makes sense since no products are being produced, so there would be no profit.

Since the goal is to find the smallest value of the objective (z) that satisfies the constraints. Since the coefficients of both x_a and x_b are negative, any increase in their value will reduce the objective. But how much can either be increased? Choosing x_b since it has the more negative coefficient.

Each constraint must remain non-negative.

$$4x_b + u \le 600 \text{ or } u \le 600 - 4x_b$$
$$3x_b + v \le 500 \text{ or } v \le 500 - 3x_b$$
$$4x_b + w \le 500 \text{ or } w \le 500 - 4x_b$$

Keep u, v and w non-negative, and limit x_b 125 (based on the third equation).

Row operations will be performed to make the coefficient of $x_b = 1$ in constraint 3 and then remove x_b from the other constraints and the objective.

$$\text{equation 3 } 2x_a + 4x_b + w = 500$$
$$\text{divide by 4 } .5x_a + x_b + .25w = 125$$

Call this equation 3A

Removing x_b from equation 1

$$\text{Equation 1} \qquad 4x_a + 4x_b + u = 600$$
$$-4 \text{ times eqn 3A } -2x_a - 4x_b - w = -500$$

$$\text{Equation 1A} \qquad 2x_a + u - w = 100$$

And from equation 2

$$\text{Equation 2} \qquad 3x_a + 2x_b + v = 500$$
$$-2 \text{ time eqn 3A } -x_a - 2x_b - .5w = -250$$

$$\text{Equation 2A} \qquad 2x_a + v - .5w = 250$$

And from the objective (equation 4)

Equation 4 $-10x_a - 12x_b = 0$

12 times eqn 3A $6x_a + 12x_b + 3w = 1500$

Equation 4A $-4x_a + 3w = 1500$

Which gives the following new simplex tableau

eqn 4A	-4	0	0	0	3	1500
eqn 1A	2	0	1	0	-1	100
eqn 2A	2	0	0	1	-0.5	250
eqn 3A	0.5	1	0	0	-0.25	125

$x_b = 125$

$u = 100$

$v = 250$

$x_a = w = 0$

And the objective is at -1500

Since the coefficient of x_a is negative we can increase it and reduce the objective further.

$2x_a + u \leq 100$ or $u \leq 100 - 2x_a$

$2x_a + v \leq 250$ or $v \leq 250 - 2x_a$

$.5x_a + x_b \leq 125$ or $x_b \leq 125 - .5x_a$

Keeping u, v and x_b non-negative, x_a is limited to 50. (first equation)

Repeating the process to make the coefficient of x_a =1 and remove it from the other constraints and objective

Equation 1A $2x_a + u - w = 100$

Divide by 2 $1x_a + .5u - .5w = 50$

Call this equation 1B

Equation 2A $2x_a + v - .5w = 250$

-2 times 1B $-2x_a - u - w = -100$

Equation 2B $-u + v - 1.5w = 150$

$$\text{Equation 3A} \quad .5x_a + x_b + .25w = 125$$

$$-.5 \text{ times 1A} \quad -.5x_a - .25u - .25w = -25$$

$$\text{Equation 3B} \quad x_b - .25w = 100$$

And the revised objective

$$\text{Equation 4A} \quad -4x_a + 3w = 1500$$

$$4 \text{ times 1A} \quad 4x_a + 2u - 2w = 200$$

$$\text{Equation 4B} \quad 2u + w = 1700$$

And a revised simplex tableau

eqn 4B	0	0	2	0	1	1700
eqn 1B	1	0	0.5	0	-1	50
eqn 2B	0	0	-1	1	-1.5	150
eqn 3B	0	1	0	0	-0.25	100

$$x_a = 50$$
$$x_b = 100$$
$$v = 150$$
$$u = w = 0$$

And the objective is at -1700. Since none of the coefficients of the variables in the objective are negative, this is the best that can be done. No lower value of the objective can be obtained. This is the optimal solution.

Checking the solution to the constraints

$$4x_a + 4x_b + u = 600$$

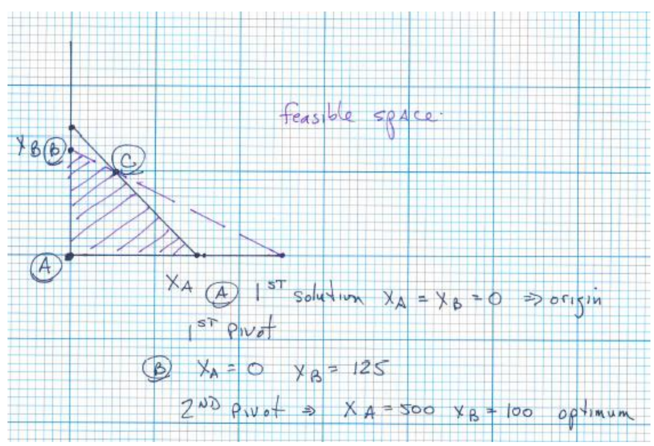
$$4(50) + 4(100) + 0 = 600$$

$$3x_a + 2x_b + v = 500$$

$$3(50) + 2(100) + 150 = 500$$

$$2x_a + 4x_b + w = 500$$

$$2(50) + 4(100) + 0 = 500$$



A graph showing the feasible region and the pivot points used for the tableau method.

Using Pivot Tables

Robert Vanderbei, Operations Research and Financial Engineering Professor at Princeton has created an online solver for the simplex method. To see how this problem would be solved using this, or a similar software solver, let's walk through the steps here.

The constraints previously were converted to the following objective function and equalities.

$$f(x_a, x_b) = 10x_a + 12x_b$$
$$4x_a + 4x_b + u = 600$$
$$3x_a + 2x_b + v = 500$$
$$2x_a + 4x_b + w = 500$$

These are entered as the dictionary shown to the right. A dictionary is much like a tableau and is pivoted repeatedly until the solution can be read from the dictionary. In the dictionary, x_1 and x_2 correspond to x_a and x_b in the previous tableaux and w_1, w_2 and w_3 correspond to u, v and w respectively.

Current Dictionary

maximize	ζ	=			10	x_1	+	12	x_2
subject to:	w_1	=	600	-	4	x_1	-	4	x_2
	w_2	=	500	-	3	x_1	-	2	x_2
	w_3	=	500	-	2	x_1	-	4	x_2

$x_1 \; x_2 \; w_1 \; w_2 \; w_3 \geq 0$

Continuing through the example, we will choose to pivot on the variable x_2 and will pivot it with w_1. The pink highlights on both x_1 and x_2 suggest that either will suffice for a pivot. We chose x_2 because it has the greater of the two coefficients and changes to it will cause a greater increase than changes to x_1. To accomplish the pivot, click on the box labelled x_2 in the line starting with w_1=. After pivoting x_2 with w_1, we get the second dictionary to the right. We still need to pivot x_1, and the tool recommends pivoting with w_3 by showing the pink highlight. Clicking on the x_1 box in the line starting with w_3= gives us the final dictionary.

Current Dictionary

maximize	ζ	=	1800	+	-2	x_1	+	-3	w_1
subject to:	x_2	=	150	-	1	x_1	-	1/4	w_1
	w_2	=	200	-	1	x_1	-	-1/2	w_1
	w_3	=	-100	-	-2	x_1	-	-1	w_1

$x_1 \; x_2 \; w_1 \; w_2 \; w_3 \geq 0$

Current Dictionary

maximize	ζ	=	1700	+	-1	w_3	+	-2	w_1
subject to:	x_2	=	100	-	1/2	w_3	-	-1/4	w_1
	w_2	=	150	-	1/2	w_3	-	-1	w_1
	x_1	=	50	-	-1/2	w_3	-	1/2	w_1

$x_1 \; x_2 \; w_1 \; w_2 \; w_3 \geq 0$

From this dictionary, remembering that the slack variables are equal to zero, we can read off the solution $x_2 = 100$ and $x_1 = 50$. This agrees with the solution we attained using the tableau method above.