Legos Optimization
A Sentry Reconnect Module

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Teacher notes are in blue. A professional development module will be available in Canvas by the end of 2023

1 Introduction

Many STEM majors would be well-suited to study optimization or, more broadly, industrial engineering, in graduate school, but most are not aware of this exciting, in-demand field. We envision Linear Algebra students as the primary target for this module because students who are studying Linear Algebra are also a likely audience for optimization, and because Linear Algebra is so fundamental to the study of optimization. We seek to introduce the field of optimization to these students through an engaging activity that both reinforces concepts from Linear Algebra and develops a conceptual framework that will be useful for students who go on to study optimization in a later course.

After a fun, hands-on activity, we introduce optimization modeling and a simple (albeit computationally expensive) algorithm for solving linear optimization models, commonly called Linear Programs or LPs, with bounded feasible regions. The algorithm relies on the basic geometry of systems of linear inequalities. The ideas in this simple algorithm are fundamental to the Simplex Method, the linear optimization algorithm that is implemented in commercial optimization solvers and that is typically taught in a first course in optimization.

1.1 Required materials

Each student will require 12 large Lego blocks (rectangular top with 8 connectors) and 18 small Lego blocks (square top with 4 connectors).

2 Audience

This module is designed with Linear Algebra students in mind. However, the module may be adapted to students with exposure to linear systems in the context of a College Algebra course by restricting the focus to problems with two variables.

2.1 Required knowledge

- Linear equations describe lines, planes, or hyperplanes (depending on the dimension
- Linear inequalities describe half-spaces bounded by lines, planes, or hyperplanes (depending on the dimension)
- Linear independence of equations
- The solution to a system of n linearly independent equations defines the point of intersection of the lines/planes/hyperplanes described by the equations

3 Objectives

I. Introduce constrained optimization through an engaging activity

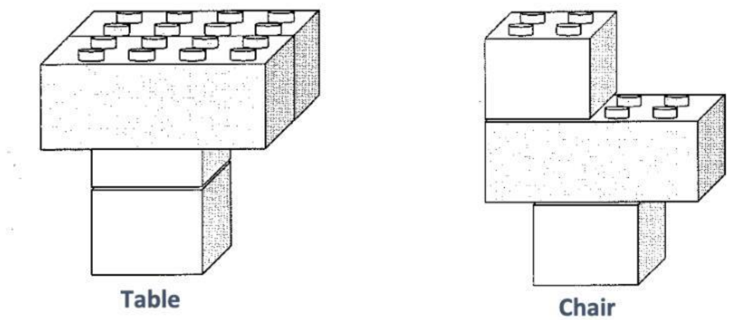
II. Introduce linear optimization modeling

III. Introduce a geometric algorithm for solving a bounded linear optimization model that (i) reinforces concepts from linear algebra, and (ii) introduces concepts that are fundamental to the study of optimization

IV. Provide a teaser for the breadth of problem-types that can be tackled using optimization

4 Legos Furniture Optimization

Problem 1. Imagine that you are running a manufacturing business that makes tables and chairs out of Legos. The tables can be sold for $16 each and the chairs are sold for $10 each. Each table requires 2 big blocks and 2 little blocks while each chair requires 1 big block and 2 little blocks, as shown below. There are 12 big and 18 small blocks available to build tables and chairs.



a. What is the maximum revenue that you can achieve with these resources?

b. How many tables would you build?

c. How many chairs would you build?

d. How do you know your solution is correct?

Solution: Students will likely begin by building 6 tables for a revenue of $96, because the revenue from a table is higher than the revenue from a chair. They will have some small blocks leftover and realize that they can take apart one table to build two chairs, increasing their revenue by $4. Doing this twice more leads to the optimal solution: 3 tables and 6 chairs for a revenue of $108.

5 Modeling the Legos Problem as a Linear Program

The Legos problem can be modeled using a linear optimization model, or linear program, a name that is often shortened to "LP". Linear programs have three main components: decision variables, an objective function, and constraints.

The decision variables in a linear program are variables that represent the decisions to be made. In this problem, we seek to determine how many tables and chairs to make. Thus, we define decision variables,

T := the number of tables to make;
C := the number of chairs to make.

Now we are ready to define our objective function: a function of the decision variables that represents the value that we wish to maximize or minimize. In this case, we wish to maximize revenue, which can be expressed as,

$$\text{Maximize Revenue} := 16T + 10C$$

Finally, we must define the constraints, which are equations or inequalities that impose real world requirements on the values that the decision variables can take. In the Legos problem, we are restricted to using no more than 12 big and 18 small blocks. Each table requires 2 big blocks and each chair requires 1 big block, so we can express the number of big blocks that we use to make our furniture as $2T + C$. Thus, we can express the limit on the number of available big blocks as,

$$2T + C \leq 12.$$

Similarly, we can express the limit on the number of small blocks that are available for our furniture making as

$$2T + 2C \leq 18.$$

The constraints limiting the number of available blocks are general constraints. Most linear programs also have variable bounds, which are a class of constraints that limit the range of the decision variables to a continuous interval of real numbers. In this case, it does not make sense to make a negative number of tables or chairs, so we add the so-called nonnegativity constraints,

$$T \geq 0, C \geq 0.$$

Now we can write the linear program (LP) for the Legos problem:

Legos LP

Decision variables:

T ≔ the number of tables to make;
C ≔ the number of chairs to make.

Model:

$$\begin{aligned} \text{Maximize} \quad & R(T,C) \coloneqq 16T + 10C \\ \text{subject to} \quad & 2T + C \leq 12 && (1) \\ & 2T + 2C \leq 18 && (2) \\ & T \geq 0 && (3) \\ & C \geq 0 && (4) \end{aligned}$$

You may be thinking, "Don't we also need to make whole numbers of tables and chairs?" This is true, of course. However, in addition to the objective function and constraints being linear, another requirement of a linear programming model is that the variables are allowed to take on any value in a continuous interval of real numbers (subject to the general constraints). A linear optimization model in which the variables are required to take on integer values is called an integer program (IP). Integer programs require a different, much more computationally expensive, algorithm. Thus, we allow fractional numbers of tables and chairs in our solutions.

The Legos model is a resource allocation linear program. Resource allocation problems typically seek to maximize profit or revenue given limited production resources such as materials, manpower, or time. Resource allocation was one of the first applications of linear programming and is still commonly used in industry.

Here are a few resource allocation problems for modeling practice. Hint: Before writing the models for these problems, ask yourself the following questions:

1. What are the decisions that need to be made in this problem? (Answering this question helps to define the decision variables.)
2. What is the goal in the problem? (Answering this question helps to define the objective function.)
3. What are the restricted resources? (This helps to define the constraints.)

Problem 2. A manufacturing company can make devices using teams composed of 2 experienced engineers or 1 experienced engineer and three interns. Experienced teams build three devices every two hours, or 12 in an eight hour shift, while trainee teams build one device per hour, or eight in an eight hour shift. If the company has 12 engineers and 18 interns allocated to making devices, how many experienced teams and trainee teams should they use to maximize the number of devices constructed?

Solution: Let x be the number of engineer-only teams and let y be the number of training teams. The model is:

Maximize	$12x + 8y$
subject to	$2x + y \leq 12$
	$3y \leq 18$
	$x \geq 0$
	$y \geq 0$

Problem 3. A company makes mp3 players to sell. They make two types of players, an economy player that sells for $45 and a higher-end player that sells for $65. The players are assembled from a battery and circuit components. One team creates the circuits and a second team makes the batteries and assembles the players. The economy model requires 1.5 worker hours for each battery and 2 worker hours for each circuit assembly. The higher end model requires 2 worker hours for each battery and 3 worker hours for each circuit assembly. In the battery assembly area there are 560 worker hours available and in the circuit assembly area there are 780 worker hours available. How many of each type of system should be built to maximize revenue?

Solution: Let x be the number of economy models produced and let y be the number of higher-end models produced. The model is:

Maximize	$45x + 65y$
subject to	$1.5x + 2y \leq 560$
	$2x + 3y \leq 780$
	$x \geq 0$
	$y \geq 0$

6 The Feasible Region

A *solution* to a linear program is an assignment of a number to each of the decision variables. For example, $(T, C) = (6,0)$ is a solution to the Legos problem that represents making six tables and no chairs.

A *feasible solution* to a linear program is a solution that satisfies all of the constraints. The solution $(T, C) = (6,0)$ to the Legos problem is a feasible solution because it satisfies all four constraints: it does not overuse large or small blocks (constraints 1 and 2), and it does not require making a negative number of tables or chairs (constraints 3 and 4).

An *infeasible solution* is a solution that is not feasible; that is, a solution is infeasible if it violates at least one constraint.

The *value* of a solution is the value of the objective function at that solution.

A constraint is *active* at a particular solution if the constraint is satisfied with equality. Constraints 1 and 4 are active for solution $(6,0)$.

Problem 4. For each of the solutions, (T, C), to the Legos problem below: (i) Determine whether the solution is feasible or infeasible. (ii) If the solution is feasible, determine which constraints in the Lego LP are active. If the solution infeasible, determine which constraints are violated. (iii) Find the value of the solution.

a. $(4,4)$
b. $(2,8)$
c. $(-1,10)$
d. $(0,9)$

Solution:

a. (i) Feasible, (ii) $2T + C \leq 12$ is active, (iii) 104

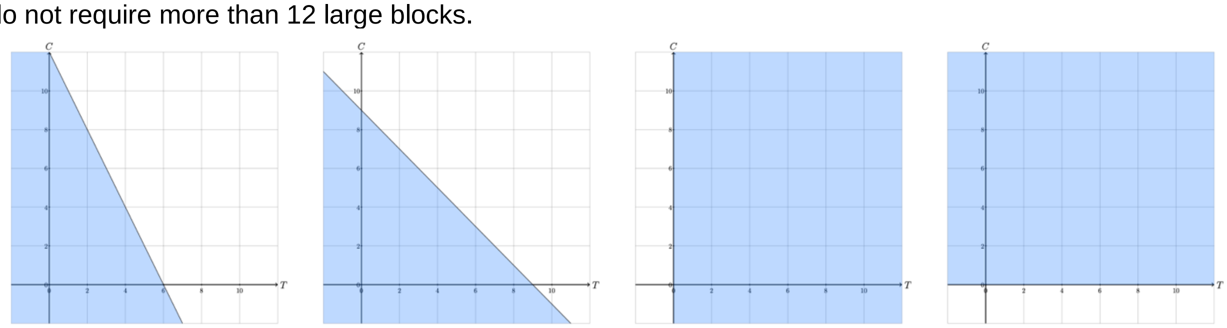
b. (i) Infeasible, (ii) $2T + 2C \leq 18$ has been violated, (iii) 112

c. (i) Infeasible, (ii) $T \geq 0$ has been violated, (iii) 84

d. (i) Feasible, (ii) $2T + 2C \leq 18$ and $T \geq 0$ are active, (iii) 90

For two-dimensional linear programs (those with two variables), we can visualize the set of points that satisfy each constraint in the Cartesian plane. A 2-dimensional linear inequality, such as $2C + T \leq 12$, describes a half-plane: all the points on the line $2C + T = 12$, along with the points to the left of (or below) this line.

Figure 1 illustrates the half-planes described by the four constraints of the Legos LP. For example, the shaded region in Figure 1a consists of all of the solutions to the Legos LP that satisfy constraint 1, or that do not require more than 12 large blocks.



(a) $2T + C \leq 12$ (b) $2T + 2C \leq 18$ (c) $T \geq 0$ (d) $C \geq 0$

Figure 1: Half-planes described by the constraints of the Legos LP

The *feasible region* of a linear program is the set of solutions that satisfy all constraints. The feasible region of the Legos LP consists of the points in the intersection of all four half-planes described by its constraints, as illustrated in Figure 1. The feasible region of the Legos LP is illustrated in Figure 2. Thus, every point in the shaded region in Figure 2 represents a valid solution to the Legos problem.

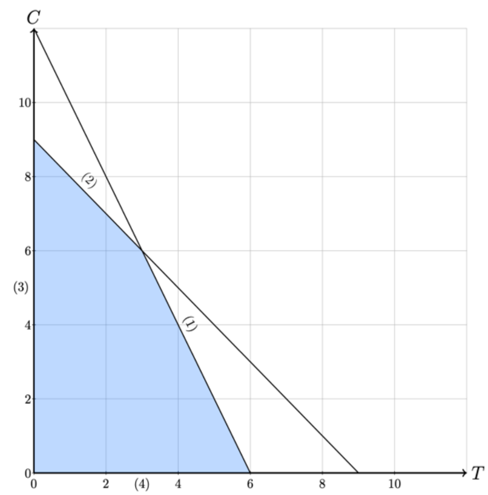


Figure 2: Feasible region of the Legos LP

A linear program is infeasible if its feasible region is empty (in other words, the linear program has no feasible solutions).

A feasible region is unbounded if one or more decision variables can increase (or decrease) indefinitely inside of the feasible region. In this case, the linear program may be unbounded, meaning that there is no optimal solution because the values of .

Problem 5. Suppose the inequalities are the constraints of a linear program. Graph the feasible region of each linear program. Is the associated linear program infeasible?

a.

$$2x + y \leq 4$$
$$x + 2y \leq 6$$
$$x \geq 0$$
$$y \geq 0$$

b.

$$-2x + y \leq -4$$
$$x + 2y \leq 1$$
$$x \geq 0$$
$$y \geq 0$$

c.

$$\begin{aligned} 2x - y &\leq -3 \\ x - 2y &\leq 6 \\ x &\geq 0 \\ y &\geq 0 \end{aligned}$$

Solution:

a. A graph of these constraints looks like Figure 3. The feasible region is the polygon with vertices $(0,0)$, $(0,3)$, $\left(\frac{2}{3},\frac{8}{3}\right)$, and $(2,0)$. Any of the points in the region or on its boundary satisfy all four constraints.

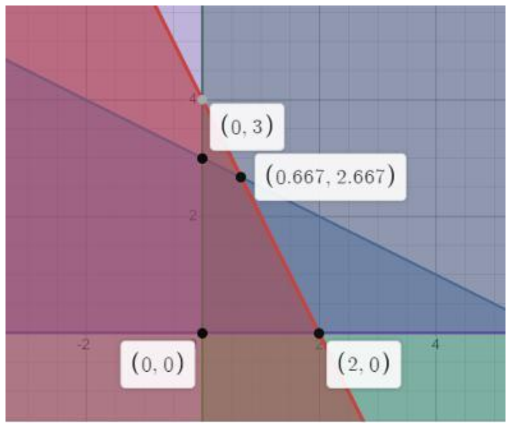


Figure 3: Feasible region: Problem 5a.

b. This is an example of a system that is infeasible. The constraints do not create a region where all four solutions overlap. The system is illustrated in Figure 4.

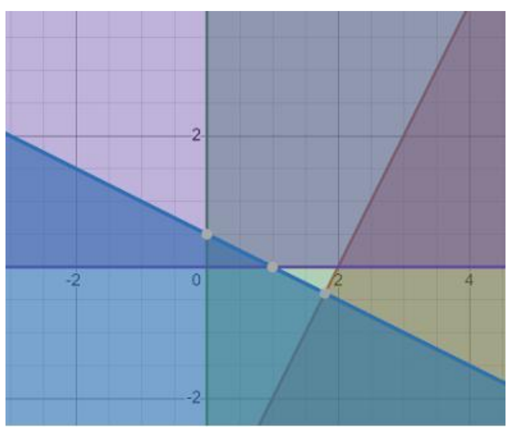


Figure 4: No feasible solutions: Problem 5b.

c. This is an example of an unbounded feasible region. The feasible region has corners at $(0,0)$, $(0,3)$, and $(6,0)$, but extends without bound between the lines $2x - y = -3$ and $x - 2y = y$ as x and y increase to infinity.

7 Solving the Legos LP

Let's consider the Legos LP again:

$$\begin{aligned} \text{Maximize} \quad & R(T,C) := 16T + 10C & \\ \text{subject to} \quad & 2T + C \le 12 & (1) \\ & 2T + 2C \le 18 & (2) \\ & T \ge 0 & (3) \\ & C \ge 0 & (4) \end{aligned}$$

The feasible region of the Legos LP is made up of the polygon with corners $(0,0)$, $(6,0)$, $(0,9)$, and $(3,6)$, as illustrated in Figure 2. These corner points are called basic feasible solutions, of the LP. These points are important in solving a linear program because at least one of these points must be an optimal solution!

> **Theorem 1.** If P is a linear program with a bounded, nonempty, feasible region, then an optimal solution to P occurs at a basic feasible solution.

Note the requirement in Theorem 1 that the feasible region of P is bounded and nonempty. Without these requirements, there are three possible cases given a linear program P : (1) P has an optimal solution, (2) P is unbounded (i.e., the objective function can grow without bound within the feasible region), or (3) P is infeasible. The requirement that the feasible region of P is bounded and nonempty keeps us in case (1), our case of interest.

Thus, we can solve the Legos problem by evaluating the objective function at each of the corner points of the feasible region to see which results in the maximum revenue:

$$R(0,0) = 0$$
$$R(0,9) = 90$$
$$R(3,6) = 108$$
$$R(6,0) = 96$$

Of the four corner points $(3,6)$, results in the maximum revenue. Thus, by Theorem 1, the optimal solution to the Legos problem is to make 3 tables and 6 chairs for a revenue of $108.

Problem 6. Use Theorem 1 to solve the LP below.

$$\begin{aligned} \text{Maximize} \quad & C(x,y) = x + y \\ \text{subject to} \quad & 7x + 2y \le 28 \\ & 2x + 2y \le 11 \\ & x \ge 0 \\ & y \ge 0 \end{aligned}$$

Solution: By graphing the feasible region of the LP, we see that the corner points are $(0,0)$, $(0,3)$, $(\frac{2}{3},\frac{8}{3})$, and $(2,0)$. Evaluating the objective at each of these points, we see that $(2,0)$ results in the maximum objective value of 12. Thus, by Theorem 1, the optimal solution is $(2,0)$.

Problem 7. Consider the following linear program:

$$\begin{aligned} \text{Maximize} \quad & C(x,y) = 6x + 2y \\ \text{subject to} \quad & 2x + 3y \leq 15 \\ & 2x + y \leq 4 \\ & x + 2y \leq 6 \\ & x \geq 0 \\ & y \geq 0 \end{aligned}$$

a) Graph the feasible region and find the corner points.

b) Solve the LP. What do you notice?

Solution: The corner points of the feasible region are $(0,0)$, $(0,5)$, $(1.5,4)$, $(3.4,2.1)$, and $(4,0)$. The two corner points $(1.5,4)$ and $(3.4,2.1)$ both result in the same maximum objective value of 5.5. In fact, all of the point on the line segment connecting $(1.5,4)$ and $(3.4,2.1)$ are optimal solutions to the LP.

8 Simple Algorithm for Solving LPs

The geometric method of solving linear programs that we used in the previous section works well when we have only two variables, but it becomes increasingly difficult to sketch the feasible region of an LP as the number of decision variables increases. However, it is possible to identify the corner points of a feasible region algebraically. Let's return the Legos problem to see how to do this.

Recall the feasible region of the Legos problem:

$$2T + C \leq 12 \quad (1)$$
$$2T + 2C \leq 18 \quad (2)$$
$$T \geq 0 \quad (3)$$
$$C \geq 0 \quad (4)$$

Suppose we replace each inequality by an equation as follows:

$$2T + C = 12 \quad (1)$$
$$2T + 2C = 18 \quad (2)$$
$$T = 0 \quad (3)$$
$$C = 0 \quad (4)$$

Each pair of linearly independent equations in the list corresponds to two lines that intersect at a single point. We can solve the corresponding two-equation sub-system to find that point of intersection. Let's try that now.

In the case of the Lego's linear program, there are six pairs of equations: (1) and (2), (1) and (3), (1) and (4), (2) and (3), and (2) and (4). Upon inspection, we see that all six pairs are linearly independent. For example, we can solve the system formed by equations (1) and (2):

$$2T + C = 12 \quad (1)$$
$$2T + 2C = 18 \quad (2)$$

to find the point of intersection of the two constraints: $(T,C) = (3,6)$. The intersection points obtained by solving all six of these 2 x 2 systems of equations are as follows:

Constraints	Intersection Point
(1) and (2)	$(3,6)$
(1) and (3)	$(0,12)$
(1) and (4)	$(6,0)$
(2) and (3)	$(0,9)$
(2) and (4)	$(9,0)$
(3) and (4)	$(0,0)$

Let's revisit the graph of the feasible region of the Legos problem to see where all of these points reside, as labeled in Figure 5.

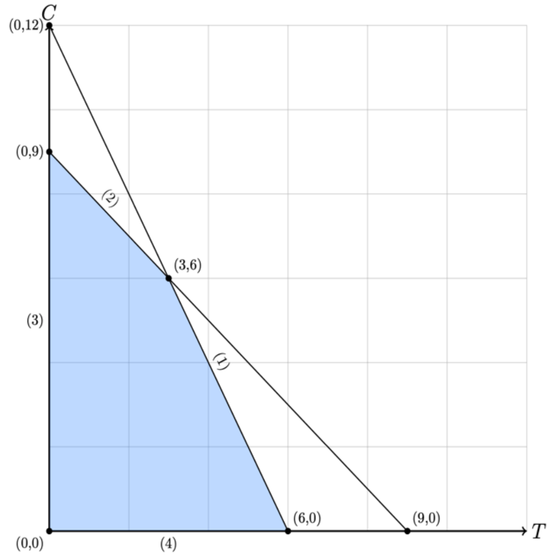


Figure 5: Feasible region of the Legos LP

Notice that our process of solving every linearly independent pair of equations resulted in all of the corner points, or basic feasible solutions, of the Legos LP, but it also resulted in two other points: $(0,12)$ and $(9,0)$. We can see in Figure 5 that these two points are at the intersections of two constraint boundaries, but they are not in the feasible region. We can see observe both geometrically and algebraically that $(9,0)$ violates constraint (1) and $(0,12)$ violates constraint (2).

The points that are found at the intersections of constraints, such as the six that we found in the table above for the Legos problem, are called *basic solutions*.

The process described above leads us to the following algorithm for solving linear programs with bounded feasible regions.

Corner Point Algorithm (to Solve an LP): Suppose linear program (P) has a bounded, nonempty feasible region. Further suppose that (P) has d decision variables. To solve (P):

i Find all sets of d linearly independent constraints of (P)

ii Solve each of the corresponding d x d linear systems, arriving at a set of basic solutions, S

iii Test each basic solution in S for feasibility, eliminating the infeasible basic solutions. The is remaining set of solutions, B, the set of basic feasible solutions, or corner points of the feasible region.

iv Evaluate the objective function of (P) at each element of B.

v Apply Theorem 1 to identify an optimal solution to (P).

Problem 8. Solve the LP using the Corner Point Algorithm described above:

$$\begin{aligned} \text{Maximize} \quad & 2x + 3y - z \\ \text{subject to} \quad & x + y + z \leq 12 \\ & x + y \leq 8 \\ & x \geq 0 \\ & y \geq 0 \\ & z \geq 0 \end{aligned}$$

Solution: The solution which maximizes the objective function is (0,8,0). The value of the objective function at that point is 24.

9 Beyond Legos

In the world of optimization, the Lego furniture problem is known as a resource allocation problem. This is one of the first applications of Linear Programming, and is still an important application in business and military settings. Linear programs have been be used to optimize: work schedules, delivery jobs and routes, airline flight routes and schedules, sports schedules, fantasy team lineups, water and gas pipelines, telecommunication networks, election polling locations, pollution mitigation strategies, and the list goes on and on and on.

During World War 2, George Danzig defined and used linear programs to develop efficient operational plans for the United States Air Force. After the war in 1947, Dantzig published the algorithm, now called the simplex method, that he had developed to efficiently solve linear programs.

Linear programming models that solve real-world problems can easily have thousands or even millions of decision variables, and billions of basic feasible solutions. Fortunately, the simplex method does not require finding every single basic feasible solution like we did. Instead, the simplex method starts at a corner point of the feasible region and traverses a path through adjacent corner points so that the objective function either stays the same or improves at each step. The simplex method terminates when none of the corner points adjacent to the current corner point improves the objective function, indicating that the current solution is optimal. The simplex method is still implemented in top open-source and proprietary linear programming solvers.